Homework for ECON 21130

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Problem Part 1 (30 points)

We consider a measurement system for a random variable of interest X which is unobserved to the econometrician. Instead we observe two variables Y_1, Y_2 related to X in the following way:

$$Y_{i1} = X_i + \epsilon_{i1}$$
$$Y_{i2} = X_i + \epsilon_{i2}$$

and we assume that $\mathbb{E}(X) = \mu$, $Var(X) = \sigma_x^2$, $X \perp \epsilon_1 \perp \epsilon_2$, $\mathbb{E}(\epsilon_1) = \mathbb{E}(\epsilon_2) = 0$ and $Var(\epsilon_1) = \sigma_1^2, Var(\epsilon_2) = \sigma_2^2$. Given a set of independent draws $\big(Y_{i1}, Y_{i2}\big)_{i=1..n}$ from our model, we consider the following estimator for μ:

$$\mu_n = \alpha \frac{1}{n} \sum_{i=1}^{n} Y_{1i} + (1-\alpha) \frac{1}{n} \sum_{i=1}^{n} Y_{2i}$$

1. (8 points) Show that μ_n is an unbiased estimator of μ.
2. (8 points) Compute the variance of μ_n as a function of α and the other parameters.
3. (8 points) Find α as a function of σ_1, σ_2 that minimizes the variance. Interpret the results.
4. (6 points) We assumed through this question that $\epsilon_1, \epsilon_2, X$ were independent. Could we find a weaker assumption and keep the unbiasedness?

Problem Part 2 (30 points)

We keep the same model as in Part 1: $Y_{ik} = X + \epsilon_{ik}$ with $\epsilon_1 \perp \epsilon_2$, $\mathbb{E}(\epsilon_k) = 0$ and $Var(\epsilon_k) = \sigma_k^2$. In this part 2, **we further assume** that the distributions of ϵ_k are normal $\epsilon_k \sim \mathcal{N}(0, \sigma_k^2)$, and that $Var(X) = 0$ such that $X = \mu$ is a constant. Under these assumptions Y_1, Y_2 are normally distributed as well.

1. (6 points) Show that under the current assumptions ($X = \mu$, $\sigma_x = 0$) we have that $Y_1 \perp Y_2$. If, as in Part 1, we had that $\sigma_x > 0$ what would $Cov(Y_1, Y_2)$ be?

2. (10 points) Given that Y_1, Y_2 are independent here, express the log-likelihood of observing $\big(Y_{i1}, Y_{i2}\big)_{i=1..n}$ given parameters μ, σ_1, σ_2[1]. Hint, it is of the following form

$$\ell_n = A_1(\sigma_1, \sigma_2) + A_2(\sigma_1) \sum_{i=1}^{n} (Y_{i1} - \mu)^2 + A_3(\sigma_2) \sum_{i=1}^{n} (Y_{i2} - \mu)^2.$$

3. (8 points) Finally, solve for the maximum likelihood estimator for μ (consider here that the variances σ_1, σ_2 are known). Explain your steps.

4. (6 points) Compare this estimator to the optimal weighting found in question (3), part 1. Interpret.

[1] use the pdf for the normal distribution with mean m and variance s^2, given by $f(x) = \frac{1}{\sqrt{2\pi s^2}} \exp\big(-\frac{(x-m)^2}{2s^2}\big)$